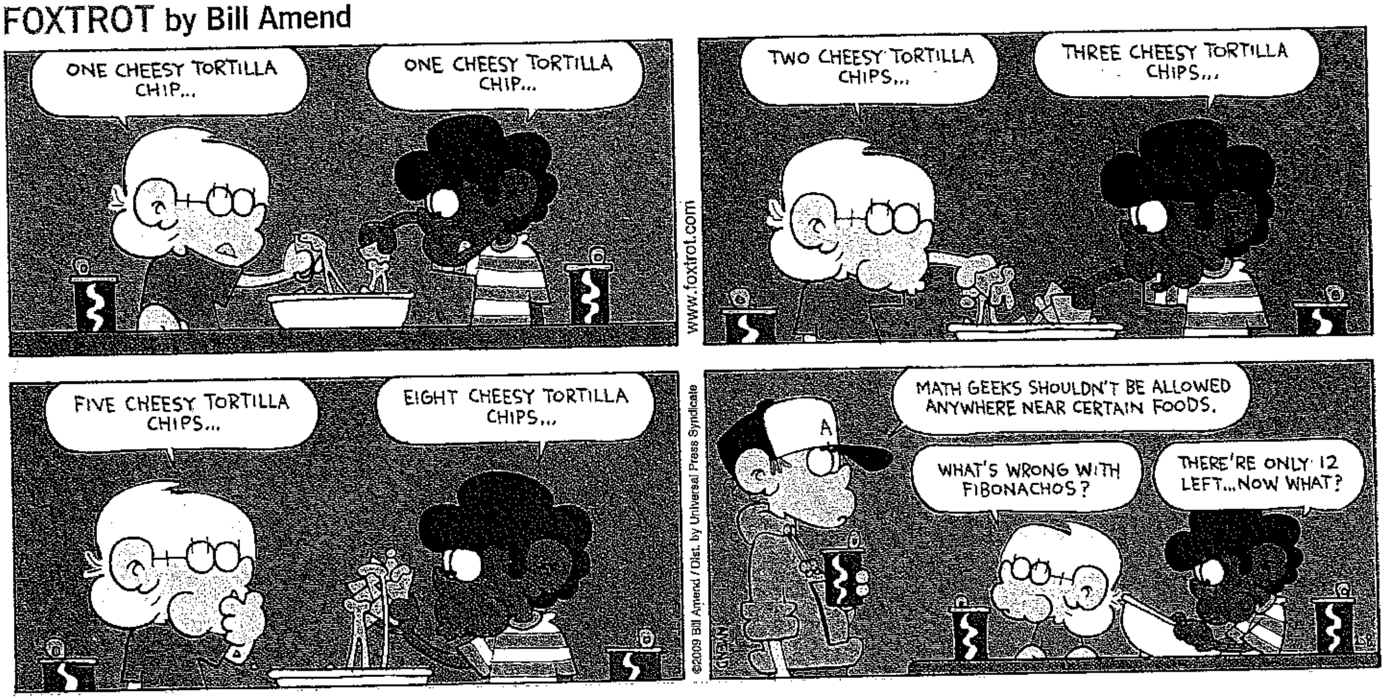
M.A.T.H. T G R

Name ______________________

FOXTROT by Bill Amend



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Here's Looking at Euclid
by Alex Bellos
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CHAPTER EIGHT

GOLD FINGER

Sitting with me in his lounge at home, Eddy Levin handed me a sheet of white paper and asked me to write out my name in capital letters. Levin, who is 75 years old and has a donnish face with gray stubble and a long forehead, used to be a dentist. He lives in north London on a street that is the epitome of prosperous and conservative suburban Britain. I took the paper and wrote: ALEX BELLOS.

Levin then picked up a stainless steel instrument that looked like a small claw with three prongs. With a steady hand he held it up to the paper and started to analyze my script. He lined up the instrument to the E in my first name with the concentration of a rabbi preparing a circumcision.

"Pretty good," he said.

Levin's claw is his own invention. The three prongs are positioned in such a way that the tips of the prongs stay on the same line and in the same ratio to one another when the claw opens out. He designed the instrument so that the distance between the middle prong and the prong above it is always 1.618 times the distance between the middle prong and the prong below it. Because this number is better known as the golden mean, he calls his tool the Golden Mean Gauge. (Other synonyms for 1.618 include the golden ratio, the divine proportion and φ, or phi.) Levin put the gauge on my letter E so that the tip of one claw was on the top horizontal bar of the E, the middle tip was on the middle bar of the E and the bottom tip was on the bottom bar. I had assumed that when I wrote a capital E I positioned the middle bar equidistant between the top and the bottom, but Levin's gauge showed that I was subconsciously placing the bar slightly above halfway—in such a way that it divided the height of the letter into two sections with lengths of ratio 1 to 1.618. Though I had scribbled my name

without any thought, I had adhered to the golden mean with uncanny precision.

Levin smiled and moved on to my S. He readjusted the gauge so that the side points touched the topmost and bottommost tips of the letter, and, to my further amazement, the middle one coincided exactly with the S line as it curved.

"Spot on," Levin said calmly. "Everybody's handwriting is in the golden proportion."

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The golden mean is the number that describes the precise ratio when a line is cut into two sections in such a way that the proportion of the entire line to the larger section is equal to the proportion of the larger section to the smaller section. In other words, when the ratio of A + B to A is equal to the ratio of A to B:

A | B

A line divided into two by the golden ratio is known as a golden section, and the ratio, phi, between larger and smaller sections can be calculated as $(1 + \sqrt{5})/2$. This is an irrational number, whose decimal expansion begins:

1.61803 39887 49894 84820 . . .

The Greeks were fascinated by phi. They discovered it in the five-pointed star, or pentagram, which was a revered symbol of the Pythagorean Brotherhood. Euclid called it the "extreme and mean ratio" and he provided a method to construct it with compass and straightedge. Since at least the Renaissance, the number has intrigued artists as well as mathematicians. The major work on the golden ratio was Luca Pacioli's *The Divine Proportion* in 1509, which listed the appearance of the number in many geometric constructions and was illustrated by Leonardo da Vinci. Pacioli concluded that the number was a message from God, a source of secret knowledge about the inner beauty of things.

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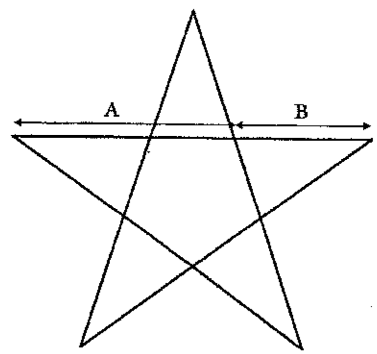
Mathematical interest in phi comes from how it is related to the most famous sequence in math: the Fibonacci sequence. This sequence starts



The pentagram, a mystical symbol since ancient times, contains the golden ratio.

with 0, 1 in which each subsequent term is the sum of the two previous terms:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377 . . .

Here is how the numbers are found:

$$0 + 1 = 1$$
$$1 + 1 = 2$$
$$1 + 2 = 3$$
$$2 + 3 = 5$$
$$3 + 5 = 8$$
$$5 + 8 = 13$$
$$\ldots$$

Before I show how phi and Fibonacci are connected, let's investigate the sequence. The natural world has a predilection for Fibonacci numbers. If you look in a garden, you will discover that for most flowers the number of petals is a Fibonacci number. The lily and the iris have three petals, the pink and the buttercup five, the delphinium eight, the marigold 13, the aster 21 and daisies either 55 or 89. The flowers may not always have these numbers of petals, but the average number of petals will be a Fibonacci number. For example, there are usually three leaves on a stem of clover, a Fibonacci number. Only seldom do clovers have four leaves, and that is

why we consider four-leaf clovers special. They are rare because 4 is not a Fibonacci number.

Fibonacci numbers also occur in the spiral arrangements on the surfaces of pinecones, pineapples, cauliflower and sunflowers. In these instances, you can count spirals clockwise and counterclockwise. The numbers of spirals you can count in both directions are consecutive Fibonacci numbers. Pineapples usually have 5 and 8 spirals, or 8 and 13 spirals. Spruce cones tend to have 8 and 13 spirals. Sunflowers can have 21 and 34, or 34 and 55 spirals—although examples as high as 144 and 233 have been found. The more seeds there are, the higher up the sequence the spirals will go.

The Fibonacci sequence is so called because the terms appear in Fibonacci's *Liber Abaci*, in a problem about rabbits. The sequence only gained the name, however, more than 600 years after the book was published when, in 1877, the number theorist Edouard Lucas was studying it, and he decided to pay tribute to Fibonacci by naming the sequence after him.

The *Liber Abaci* set up the sequence like this: Say that you have a pair of rabbits, and after one month, the pair gives birth to another pair. If every adult pair of rabbits gives birth to a pair of baby rabbits every month, and it takes one month for the baby rabbits to become adults, how many rabbits are produced from the first pair in a year?

The answer is found by counting rabbits month by month. In the first month, there is just one pair. In the second there are two, as the original pair has given birth to a pair. In the third month there are three, since the original pair has again bred, but the first pair are only just adults. In the fourth month the two adult pairs breed, adding two to the population of three. The Fibonacci sequence is the month-on-month total of pairs:

	Total pairs
First month: 1 adult pair	1
Second month: 1 adult pair and 1 baby pair	2
Third month: 2 adult pairs and 1 baby pair	3
Fourth month: 3 adult pairs and 2 baby pairs	5
Fifth month: 5 adult pairs and 3 baby pairs	8
Sixth month: 8 adult pairs and 5 baby pairs	13

An important feature of the Fibonacci sequence is that it is *recurrent*, which means that each new term is generated by the values of previous terms. This helps explain why the Fibonacci numbers are so prevalent in natural systems. Many life-forms grow by a process of recurrence.

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In addition to its association with fruit and promiscuous rodents, the Fibonacci sequence has many absorbing mathematical properties. Listing the first 20 numbers will help us see the patterns. Each Fibonacci number is traditionally written using an F with a subscript to denote the position of that number in the sequence:

F_0	0						
F_1	1	F_6	8	F_{11}	89	F_{16}	987
F_2	1	F_7	13	F_{12}	144	F_{17}	1597
F_3	2	F_8	21	F_{13}	233	F_{18}	2584
F_4	3	F_9	34	F_{14}	377	F_{19}	4181
F_5	5	F_{10}	55	F_{15}	610	F_{20}	6765

Upon closer examination, we see that the sequence regenerates itself in many surprising ways. Look at F_3, F_6, F_9, in other words, every third F-number. They are all divisible by 2. Compare this with F_4, F_8, F_{12}, or every fourth F-number—they are all divisible by 3. Every fifth F-number is divisible by 5, every sixth F-number divisible by 8, and every seventh number by 13. The divisors are precisely the F-numbers in sequence.

Another amazing example comes from $1/F_{11}$, or 1/89. This number is equal to the sum of

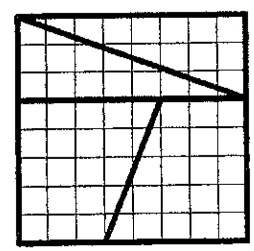
.0
.01
.001
.0002
.00003
.000005
.0000008
.00000013
.000000021
.0000000034

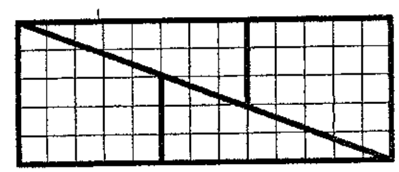
So, the Fibonacci sequence pops its head up again.

Here's another interesting mathematical property of the sequence. Take any three consecutive F-numbers. The first number multiplied by the third number is always different by 1 from the second number squared:

For F_4, F_5, F_6:
$F_4 \times F_6 = F_5 \times F_5 - 1$ $(24 = 25 - 1)$
For F_5, F_6, F_7:
$F_5 \times F_7 = F_6 \times F_6 + 1$ $(65 = 64 + 1)$
For F_{18}, F_{19}, F_{20}:
$F_{18} \times F_{20} = F_{19} \times F_{19} - 1$ $(17{,}480{,}760 = 17{,}480{,}761 - 1)$

This property is the basis of a centuries-old magic trick, in which it is possible to cut up a square of 64 unit squares into four pieces and reassemble them to make a rectangle of 65 pieces. Here's how it's done: draw a square of 64 unit squares. It has a side length of 8. In the sequence, the two F-numbers preceding 8 are 5 and 3. Divide the square up using the lengths of 5 and 3. The pieces can be reassembled to make a rectangle with sides the length of 5 and 13, which has an area of 65:


8 × 8 = 64


5 × 13 = 65

The trick is explained by the fact that the shapes are not a perfect fit. Though it is not that obvious to the naked eye, there is a long thin gap along the middle diagonal with area of one unit.

In the early seventeenth century, the German astronomer Johannes Kepler wrote that "as 5 is to 8, so 8 is to 13, approximately, and as 8 is to 13, so 13 is to 21, approximately." In other words, he noticed that the ratios of consecutive F-numbers were similar. A century later the Scottish mathematician Robert Simson saw something even more incredible. If you take the ratios of consecutive F-numbers, and put them in the sequence

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8}, \frac{21}{13}, \frac{34}{21}, \frac{55}{34} \ldots$$

or (to three decimal places)

1, 2, 1.5, 1.667, 1.6, 1.625, 1.615, 1.619, 1.618 . . .

the values of these terms get closer and closer to phi, the golden ratio.

In other words, the golden ratio is approximated by the ratio of consecutive Fibonacci numbers, with the approximation increasing in accuracy further down the sequence.

Now let's continue with this line of thought and consider a Fibonacci-like sequence, starting with two random numbers, and then adding consecutive terms to continue the sequence. So, say we start with 4 and 10; the following term will be 14 and the one after that 24. Our example gives us:

4, 10, 14, 24, 38, 62, 100, 162, 262, 424 . . .

Look at the ratios of consecutive terms:

$$\frac{10}{4}, \frac{14}{10}, \frac{24}{14}, \frac{38}{24}, \frac{62}{38}, \frac{100}{62}, \frac{162}{100}, \frac{262}{162}, \frac{424}{262} \ldots$$

or

2.5, 1.4, 1.714, 1.583, 1.632, 1.612, 1.620, 1.617, 1.618 . . .

The Fibonacci recurrence algorithm of adding two consecutive terms in a sequence to make the next one is so powerful that *whatever* two numbers you start with, the ratio of consecutive terms always converges to phi. I find this a totally enthralling mathematical phenomenon.

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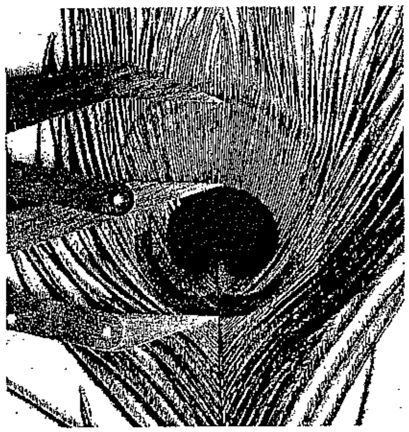
The ubiquity of Fibonacci numbers in nature means that phi is also ever-present in the world. This brings us back to the retired dentist, Eddy Levin. Early in his career he spent a lot of time making false teeth, which

he found a very frustrating job, because no matter how he arranged the teeth he could not make a person's smile look right. "I sweated blood and tears," he said. "Whatever I did the teeth looked artificial." But at around that time Levin started attending a math and spirituality class, where he learned about phi. Levin was made aware of Pacioli's *The Divine Proportion* and was inspired. What if phi, which Pacioli claimed revealed true beauty, also held the secret of divine dentures? It was 2 A.M. and Levin rushed to his study. "I spent the rest of the night measuring teeth." Levin scoured through photographs and discovered that in the most attractive sets of teeth, the big top front tooth (the central incisor) was wider than the one next to it (the lateral incisor) by a factor of phi. The lateral incisor was also wider than the adjacent tooth (the canine) by a factor of phi. And the canine was wider than the one next to it (the first premolar) by a factor of phi. Levin was measuring not the size of actual teeth, but the size of pictures of teeth when taken head-on. Still, he felt he had made a historic discovery: the beauty of a perfect smile was prescribed by phi.

"I was very excited," remembered Levin. At work, he mentioned his findings to colleagues, but they dismissed him as an oddball. He continued to develop his ideas nonetheless, and, in 1978, he published an article expounding them in the *Journal of Prosthetic Dentistry.* "From then, people got interested in it," he said. "Now there is not a lecture that is given on [dental] aesthetics that doesn't include a section on the golden proportion." Levin was using phi so much in his work that in the early 1980s he asked an engineer to design him an instrument that could tell him if two teeth were in the golden proportion. The result was the three-pronged Golden Mean Gauge. He still sells it to dentists around the world.

Levin told me his gauge became more than a work tool, and he started to measure objects other than teeth. He found phi in the patterns of flowers, in the spread of branches along stems, and in leaves along branches. He took it with him on holiday and found phi in the proportions of buildings. He also found phi in the rest of the human body: in the length of knuckles to fingers and in the relative positions of the nose, teeth and chin. Additionally, he noticed that most people use phi in their handwriting, just as he had shown in mine.

The more Levin looked for phi, the more he found it. "I found so many coincidences, I started to wonder what it was all about." He opened his laptop and showed me slides of images, each with the three points of the gauge showing exactly where the ratio was to be found. I saw pictures of butterfly wings, peacock feathers and animal colorings, the ECG reading of a healthy human heart, paintings by Mondrian and a car.



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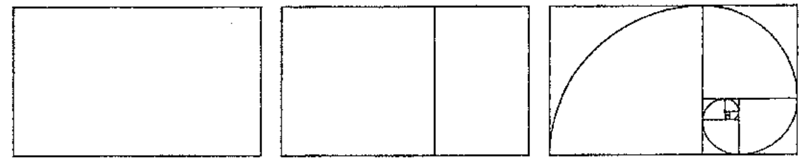
When a rectangle is constructed so that the ratio between its sides is phi, you get what is known as a "golden rectangle," as shown opposite top left. This rectangle has the convenient property that if we were to cut it vertically so that one side is a square, then the other side is also a golden rectangle. The mother gives birth to a baby daughter. If you continue this process you create granddaughters, great-granddaughters, and so on, ad infinitum. Now, let's draw a quarter circle in the largest square by using a compass, placing the point at the bottom right corner and moving the pencil from one adjacent corner to the other. Repeat in the second largest square with the compass point at the bottom left corner, with the pencil continuing the curve for another quarter circle, and then carry on with the smaller squares. The curve is an approximation of a *logarithmic spiral.*

A true logarithmic spiral will pass through the same corners of the same squares, yet it will wind itself smoothly, unlike the curve in the diagram, which will have small jumps in curvature where the sections of the quarter circle meet. In a logarithmic spiral, a straight line from the center of the spiral—the "pole"—will cut the spiral curve at the same angle at all points, which is why Descartes called the logarithmic spiral an "equiangular spiral."

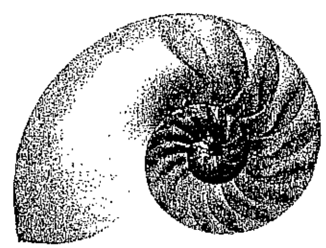


Golden rectangle and logarithmic spiral.

The logarithmic spiral is one of the most bewitching curves in math. In the seventeenth century, Jakob Bernoulli was the first mathematician to thoroughly investigate its properties. He called it the *spira mirabilis,* the wonderful spiral. He asked to have one engraved on his tombstone, but the sculptor engraved a different sort of spiral by mistake.

The fundamental property of the logarithmic spiral is no matter how much it grows, it never changes shape. Bernoulli expressed this on his tombstone with the epitaph *Eadem mutata resurgo,* or "Although changed, I shall arise the same." The spiral rotates an infinite number of times before reaching its pole. If you took a microscope and looked at the center of a logarithmic spiral you would see the same shape that you would see if the logarithmic spiral on this page were continued until it was as big as a galaxy and you were looking at it from a different solar system. In fact, many galaxies are in the shape of logarithmic spirals. Just like a fractal, a logarithmic spiral is self-similar: that is, any smaller piece of a larger spiral is identical in shape to the larger piece.

The most stunning example of a logarithmic spiral in nature is the nautilus shell. As the shell grows, each successive chamber is larger, but has the same shape as the chamber before. The only spiral that can accommodate chambers of different sizes with the same relative dimensions is Bernoulli's *spira mirabilis.*



Nautilus shell.

As Descartes noted, a straight line from the pole of a logarithmic spiral always cuts the curve at the same angle, and this feature explains why the spiral is used by peregrine falcons when they attack their prey. Peregrines do not swoop in a straight line, but rather bear down on prey by spiraling around it. In 2000, Vance Tucker of Duke University figured out why this is so. Falcons have eyes at the sides of their heads, so if they want to look in front of themselves, they need to turn the head 40 degrees. Vance tested falcons in a wind tunnel and showed that with the head at such an angle, the wind drag on a falcon is 50 percent greater than it would be if the falcon was looking straight ahead. The path that lets the bird keep its head in the most aerodynamic position possible, while also enabling it to constantly look at the prey at the same angle, is a logarithmic spiral.

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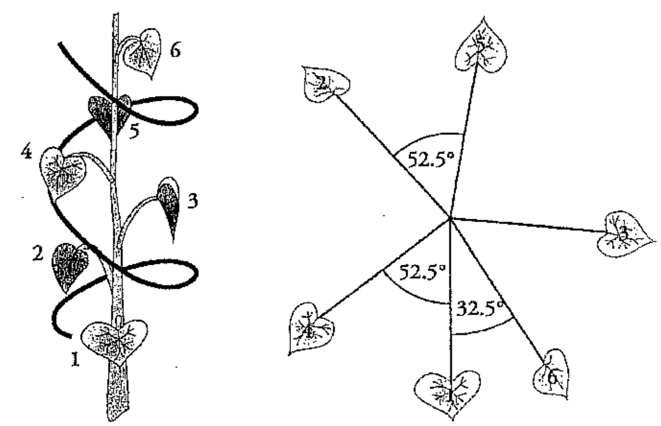
When a plant grows, it needs to position its leaves around the stem in such a way as to maximize the amount of sunlight that falls on each leaf. That's why plant leaves aren't directly above each other; if they were, the bottom ones would get no sunlight at all.

As the stem goes higher, each new leaf appears at a fixed angle around the stem from the previous leaf. The stem sprouts a leaf at a predetermined rotation. What is the fixed angle that maximizes sunlight for the leaves, the angle that will spread out the leaves around the stem so that they overlap as little as possible? The angle is not 180 degrees, or a half turn, because the third leaf would be directly above the first. The angle is not 90 degrees, or a quarter turn, because if this were the case, the fifth leaf would be directly over the first—and also, the first three leaves would be using only one side of the stem; this would be a waste of the sunlight available on the other side. The angle that provides the best arrangement is 137.5 degrees, and the diagram opposite shows where the leaves will be positioned if successive leaves are always separated by this angle. The first three leaves are positioned well apart from one another. The next two, leaves four and five, are separated by more than 50 degrees from their nearest leaves, an angle that still gives them a good amount of room. The sixth leaf is at 32.5 degrees from the first. This is closer to a leaf than any previous one, which it has to be since there are more leaves, yet the distance is still a pretty wide berth.

The angle of 137.5 degrees is known as the golden angle. It is the angle we get when we divide the full rotation of a circle according to the golden



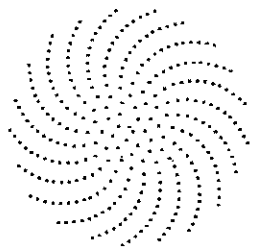
How leaves spiral up a stem.

ratio. In other words, when we divide 360 degrees into two angles such that the ratio of the larger angle to the smaller angle is phi, or 1.618. The two angles are 222.5 degrees and 137.5 degrees, to one decimal place. The smaller one is known as the golden angle.

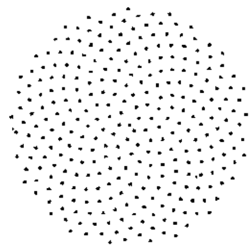
The mathematical reason why the golden angle produces the best leaf arrangement around a stem is linked to the concept of irrational numbers, which are those numbers that cannot be expressed as fractions. If an angle is an irrational number, no matter how many times you turn it around a circle you will never get back to where you started. It may sound Orwellian, but some irrational numbers are more irrational than others. And no number is more irrational than the golden ratio. (There's a brief explanation why in Appendix 6 at the website for this book.)

The golden angle explains why you generally find that on a plant stem, the number of leaves and number of turns before a leaf sprouts more or less directly above the first one is a Fibonacci number. For example, roses have 5 leaves every 2 turns, asters have 8 leaves for every 3 turns and almond trees have 13 leaves for every 5 turns. Fibonacci numbers occur because they provide the nearest whole-number ratios for the golden angle. If a plant sprouts 8 leaves for every 3 turns, each leaf occurs every ⅜ turn, or every 135 degrees, a very good approximation of the golden angle.

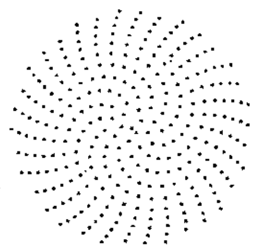
The unique properties of the golden angle are most strikingly seen in seed arrangements. Imagine that a flower head produces seeds from the center point at a fixed angle of rotation. When new seeds emerge, they push the older seeds further out from the center. The following three diagrams show the patterns of seeds that emerge with three different fixed angles: just below the golden angle, the golden angle, and just above.



Angle = 137.3 degrees
Just under golden angle



Angle = 137.5 degrees
The golden angle



Angle = 137.7 degrees
Just over golden angle

What is surprising is how a tiny change in the angle can cause such a huge variation in the positions of the seeds. At the golden angle, the seed head is a mesmerizing pattern of interlocking logarithmic spirals. It is the most compact arrangement possible. Nature chooses the golden angle because of this compactness—the seeds are bound together more closely and the organism will be stronger because of it.

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In the late nineteenth century the German Adolf Zeising most forcefully put forth the view that the golden proportion is beauty incarnate, describing the ratio as a universal law "which permeates, as a paramount spiritual ideal, all structures, forms and proportions, whether cosmic or individual, organic or inorganic, acoustic or optical; which finds its fullest realization, however, in the human form." Zeising was the first person to claim that the front of the Parthenon is in the shape of a golden rectangle. In fact, there is no documentary evidence that those in charge of the architectural project, who included the sculptor Phidias, used the golden ratio. Nor, if you look closely, is the golden rectangle a precise fit. The edges of the pedestal fall outside. Yet it was Phidias's connection to the Parthenon that, around 1909, inspired the American mathematician Mark Barr to name the golden ratio phi.

Despite the eccentric tone of Zeising's work, he was taken seriously by

Gustav Fechner, one of the founders of experimental psychology. In order to discover if there was any empirical evidence that humans thought the golden rectangle more beautiful than any other sort of rectangle, Fechner devised a test in which subjects were shown a number of different rectangles and asked which they preferred.

Fechner's results appeared to vindicate Zeising. The rectangle closest to a golden one was the top choice, favored by just over a third of the sample group. Even though Fechner's methods were crude, his rectangle testing began a new scientific field—the experimental psychology of art—as well as the narrower discipline of "rectangle aesthetics." Many psychologists have conducted similar surveys on the attractiveness of rectangles, which is not as absurd as it sounds. If there were a "sexiest" rectangle, this shape would be of use to the designers of commercial products. Indeed, credit cards, cigarette packets, and books often approach the proportions of a golden rectangle. Unfortunately for phi-philes, the most recent and detailed piece of research, by a team led by Chris McManus of University College London, suggests that Fechner was wrong. The 2008 paper states that "more than a century of experimental work has suggested that the golden section actually plays little normative role in subjects' preferences for rectangles." Yet the authors did not conclude that analyzing rectangle preference is a waste of time. Far from it. They claimed that while no one rectangle is universally preferred by humans, there are important individual differences in the aesthetic appreciation of rectangles that merit further investigation.

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Gary Meisner is a 53-year-old business consultant from Tennessee. He calls himself the Phi Guy and sells merchandise on his website including phi T-shirts and mugs. His best-selling product, however, is the Phi-Matrix, a piece of software that creates a grid on your computer screen to check images for the golden ratio. Most purchasers use it as a design tool, to make cutlery, furniture and homes. Some customers use it for financial speculation by superimposing the grid on graphs of indexes, and using phi to predict future trends. "A guy in the Caribbean was using my matrix to trade in oil, a guy in China was using it to trade in currencies," he said. Meisner was drawn to the golden mean because he is spiritual and says it helped him understand the universe, but even the Phi Guy thinks that his fellow travelers can go too far. He is, for example, unconvinced by the traders. "When you look back on the market it is pretty easy to find relationships that conform to phi," he said. "The challenge is that looking backwards is completely different from looking out the front window." Meisner's website has made him the go-to guy for every flavor of phi aficionado. He told me that a month ago he received an e-mail from an unemployed man who believed that the only way to get a job interview was to design his résumé in the proportions of the golden ratio. Meisner felt the man was deluded and took pity on him. He gave him some phi design tips, but suggested that it would be more fruitful investing in more traditional job-hunting methods like business networking. "I got a letter from him this morning," Meisner blurted. "He said he has a job interview. He is giving credit to the résumé's new design!"

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Back in London I told Eddy Levin the story of the golden résumé as an example of excessive eccentricity. Levin, however, didn't think it was funny. In fact, he agreed that a phi-proportioned résumé was better than a regular one. "It would look more beautiful, and so the reader would be more attracted to it."

After 30 years of studying the golden ratio, Levin is convinced that wherever there is beauty, there will be phi. "Any art which looks good, the dominant proportions are the golden proportion," he said. He knows this is an unpopular viewpoint, as it prescribes a formula for beauty, but he guarantees he will be able to find phi in any piece of art.

My instinctive reaction to Levin's phi obsession was one of skepticism. For a start, I was unconvinced that his gauge was accurate enough to measure 1.618 sufficiently precisely. It was not surprising to find a ratio of "approximately phi" in a painting or a building, especially if you could select which parts to choose. Also, since the ratio of consecutive Fibonacci numbers makes a good approximation to 1.618, whenever there is a grid of 5×3 or 8×5 or 13×8 and so on you will see a golden rectangle. Of course the ratio will be a common one.

Yet there was something compelling about Levin's examples. I felt the thrill of wonder with each new image he showed me. Phi really was everywhere. Yes, the golden ratio has always attracted cranks, but this in itself did not mean that all the theories were crankish. Some very respectable academics have claimed that phi creates beauty, particularly in the structure of musical compositions. The argument that human beings might be

drawn to a proportion that best expresses natural growth and regeneration does not seem too far-fetched.

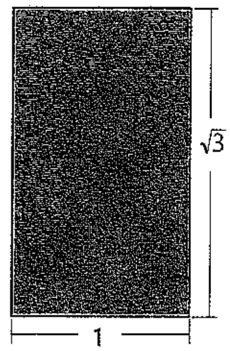
It was a sunny summer's day and Levin and I relocated to his garden. We sat on two lawn chairs and sipped tea. Levin told me that the limerick was a successful form of poetry because the syllables in its lines (8, 8, 5, 5, 8) are Fibonacci numbers. Then I had an idea. I asked Levin if he knew what an iPod was. He didn't. I had one in my pocket and I took it out. It was a beautiful object, I said, and according to his reasoning, it should contain the golden ratio.

Levin took my shiny white iPod and held it in his palm. Yes, he replied, it was beautiful, and it should. Not wanting to get my hopes up, he warned me that factory-produced objects often do not follow the golden ratio perfectly. "The shape shifts slightly for the convenience of manufacture," he said.

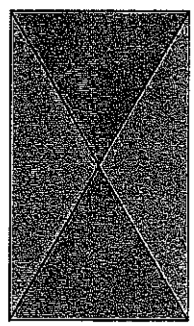
Levin opened his calipers and started measuring between all the significant points.

"Ooh, yes." He grinned.

Bronze Rectangle

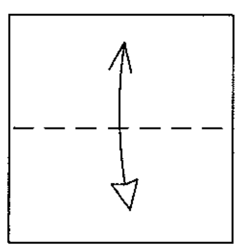


The bronze rectangle has sides proportional to $1 \times \sqrt{3}$.



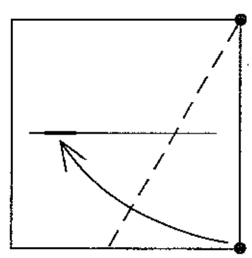
The diagonals highlight two equilateral triangles.

1



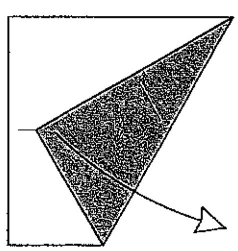
Fold and unfold.

2



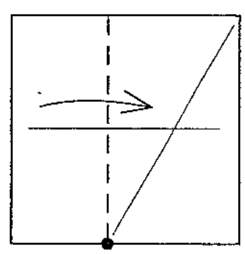
Bring the corner to the crease.

3



Unfold.

4



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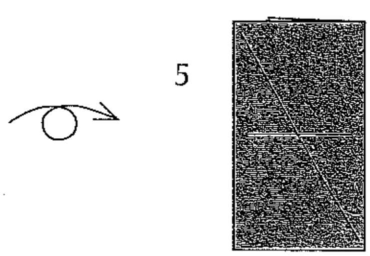
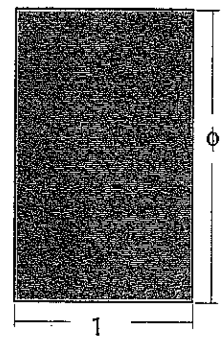


Bronze Rectangle

Golden Rectangle



The golden rectangle has sides proportional to 1×1.618034. This is the same as $.618034 \times 1$.

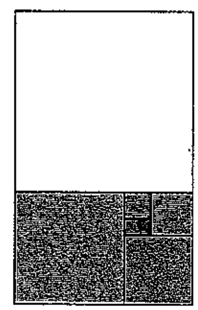
The name comes from the golden mean (phi = ϕ) where

$$\phi = \frac{\sqrt{5}+1}{2} = 1.618034$$

It is the solution to

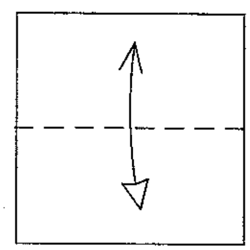
$$x - 1 = 1/x$$
$$\phi - 1 \approx .618034$$

This number is associated with nature and beauty.



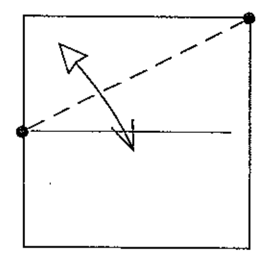
The golden rectangle divides into a square and a smaller golden rectangle.

1



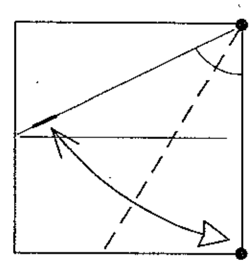
Fold and unfold.

2



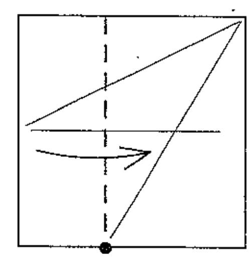
Fold and unfold.

3



Fold and unfold.

4



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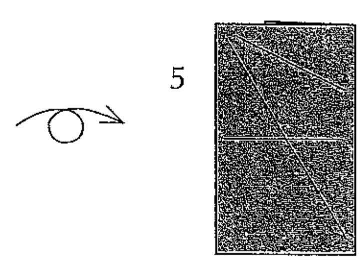
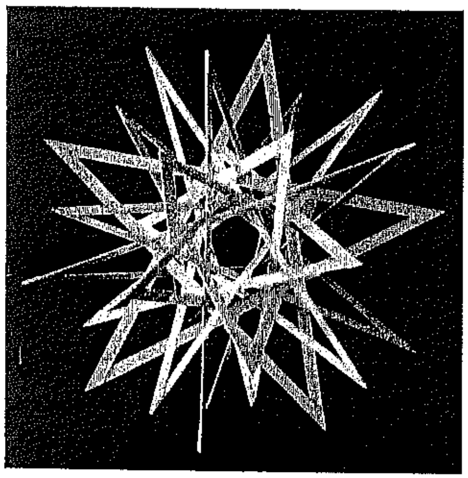


Golden Rectangle

Starcage

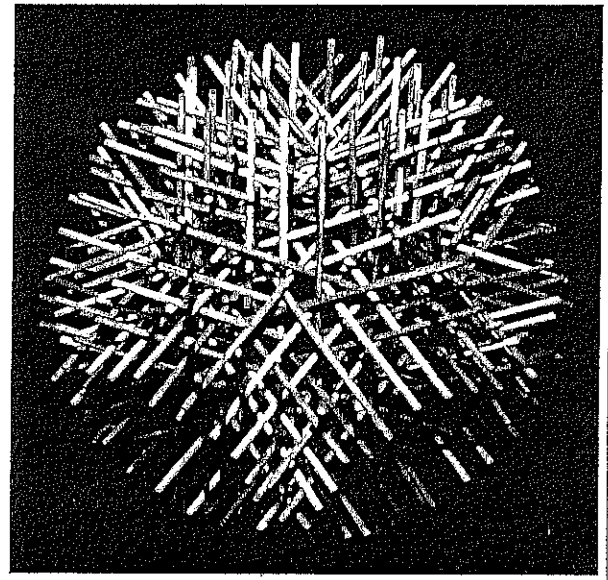
and the works of Akio Hizume

What do the mathematical concepts of lattice theory, Penrose tiling, the golden mean, the Fibonacci sequence, pentagonal symmetries and quasi-crystal geometry have in common? They are major players in the works of Japanese architect Akio Hizume. His genius and imagination combine architecture and mathematics to create exciting new shapes which reflect his fascination with structures found in mathematics and nature. As he says, "I don't separate both science and art. Both are human arts."[1] As a result, his sculptures, architecture, and music evolve from these mathematical ideas.



The Starcage, PLEIADES ©1995 Akio Hizume. It consists of 6 pentagrams and 30 plastic rods. It is commercially available at starcage@mbb.nifty.com)

Imagine a group of congruent pentagrams, each made from 5 rods, held together without the use of any adhesive, wires or strings, but by the tension created by their interwoven parts. The pentagrams lie flat against one another until they are hit or tossed onto a flat surface. Then, as if by magic, a 3-dimensional geometric shape emerges. In 1999, Hizume created a Starcage[2] consisting of 180 rods. His design won the



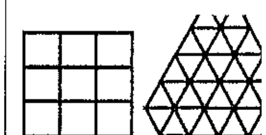
Starcage: MU-MAGARI No. 5 © 1999 Akio Hizume. It consists of 180 rods also designed around the symmetry of a dodecahedron using quasi-periodic patterns.

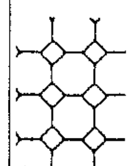
Silver Prize at the International Design Competition in Osaka, Japan. His MU-MAGARI[3] Starcage also consists of 180 rods and is designed around the symmetry of a dodecahedron using quasi-periodic patterns. All his Starcages are totally self-supporting. In fact, he has even created a self-standing Starcage (BAMBOO HENGE No. 5), which allows people to enter into its center. Hizume uses bamboo rods to make his Starcages.

Utilizing the golden mean and Fibonacci numbers, Hizume composed Fibonacci Kecak—music consisting of only 9 periodic rhythms, which repeat every 2000 years! (You can hear a 7 minute clip from his piece by logging onto: http://homepage1.nifty.com/starcage/fibonaccikck.html)

A *lattice* is an infinite array of points, spaced so that any point can be shifted onto any other point in the lattice by the arrangement of symmetry. The points defined by the x,y-coordinates of integers in a Cartesian plane is an example of a lattice. Among other types of lattices are those found in crystallography.

Tiling in mathematics is also called tessellating, which is covering a plane with a particular shape or shapes so that no gaps are left. The diagrams show how congruent equilateral triangles, squares or hexagons can be used to tile a plane such as a floor. The vertices match perfectly leaving no holes.

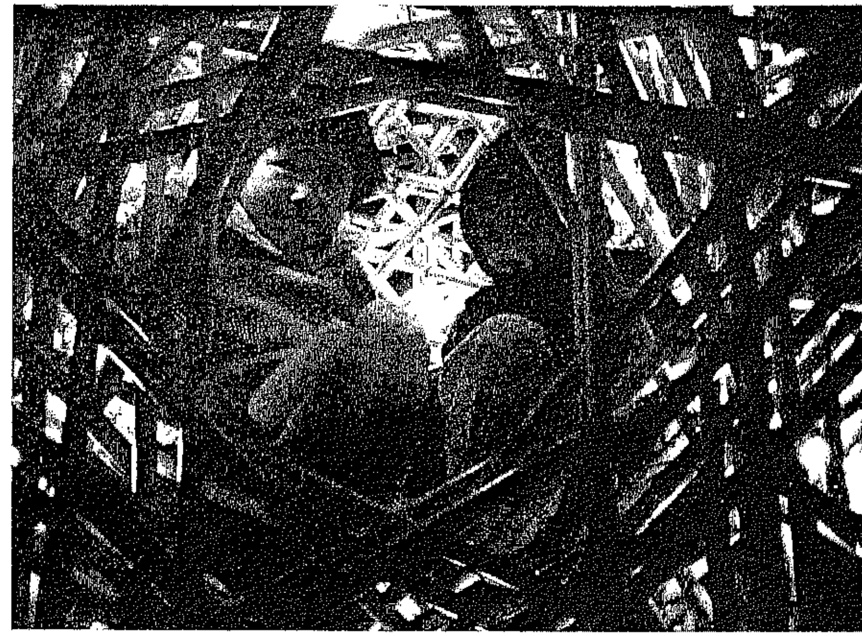




Here is how squares and octagons can cover a floor.

In 1997 he was commissioned[4] to design the *Democracy Steps* for Cedar Falls, Ohio. He specifically designed the descending pathway of steps, which reflects mathematical principles of the Fibonacci sequence and a one-dimensional Penrose lattice, so they would lead to one of Ohio's most beautiful waterfalls. The individual steps are varied so that the walker alternates the leading foot and establishes a comfortable rhythm. Hizume designed *Democracy Steps* to be as effortless as possible, thereby making it feasible for almost any walker to experience art in a public space. In addition, *Democracy Steps* lets the walker focus on and enjoy the natural surroundings of the walk rather than having to concentrate on the effort or steps of the walk.

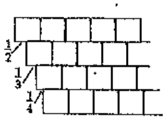
Democracy Steps. ©1997 Akio Hizume



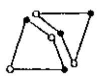
Hizume's BAMBOO HENGE No.5. © 1998 Akio Hizume. *Nakano, Tokyo, Japan. Photo: T. Ninomiya.*

In his works of "neuro-architecture", Hizume draws on such mathematical ideas as Penrose tiling. His design illustrates an experimental city. Hizume feels "*there is an essential power in architecture to educate people and to create more freedom in and for them. Many museums are rectilinear, with square rooms, and exhibits are arranged chronologically. However, in neuro-architecture, linear paths do not exist; people can access its spaces randomly. They may, at first, become confused and perhaps even get lost within neuro-architecture, but*

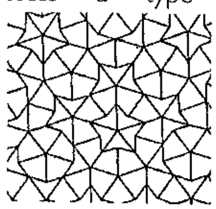
These examples are known as *regular periodic tilings*, here the design repeats on a regular basis as the eye moves vertically or horizontally. A pattern is not repeated in *non-periodic tilings*. For example, consider tiling with staggering squares.



One of the most famous nonperiodic set of tiles is the *Penrose tiles*, composed of just two shapes, a dart and a kite.



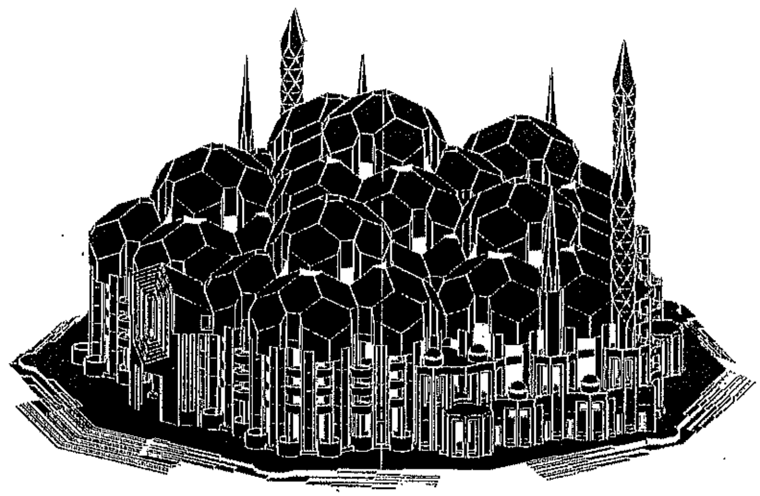
Penrose tiles possess a type of



symmetry called *fivefold (rotational) symmetry* which means a tiling pattern can be matched up to another on the plane after it is rotated 1/5 of the way around, as can be done with a pentagram. Penrose tiles also have *tenfold symmetry*.

Two flat objects are symmetrical to one another if they can be made to match up when they are either reflected about a line, rotated about a point, or translated (moved) or glided in particular direction.

Quasi-crystals were discovered in 1982. Until this time, all crystals were considered periodic, i.e. composed of a periodic arrangement of identical polyhedron building blocks, and were considered a 3-D periodic tiling. In 1982 chemist Daniel Shechtmann found a way to produce a crystal that did not



Goetheanum 3 axonometric projections, exterior. ©1990 Akio Hizume. ink on paper 415x580 mm.

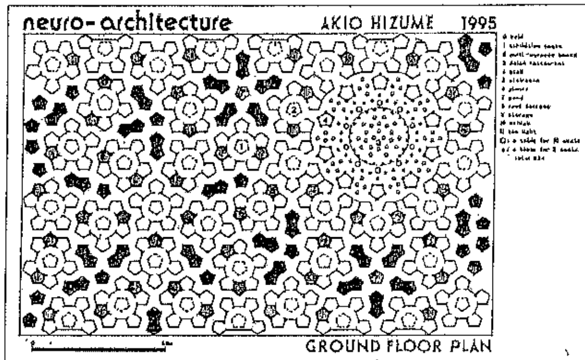
have 3,4, or 6-fold symmetry periodic tiling. In 1984, physicist Paul Steinhardt verified that these nonperiodic crystals possessed 5-fold symmetry, and he called them quasi-crystals.

The *golden mean* (also called the golden ratio and golden section) is the point on a line segment, A____B__C creating the following ratio (AC/AB) =(AB/BC). The golden mean appears in many shapes. Among the most popular shapes in which it appears are the golden rectangle (a golden rectangle can then be formed with side AC and AB) and the pentagram. The *Fibonacci numbers* 1,1,2,3,5,8,13,... first appeared as a solution to a

as they become more familiar with it, their minds will become educated and more advanced....In a sense, neuro-architecture is a two-dimensional arrangement of the one-dimensional Democracy Steps"[5]. As he points out, Penrose lattices appear in nature so why not in architectural designs. Utilizing one, two, and three-dimensional Penrose-lattices as a grid planning, he refers to them as a *"self-similar and quasi-periodic city"* and the *Goetheanum 3* monument as a "six dimensional" structure because *"The architecture was designed based on six equivalent coordinate axes. It seems to be very complicated in a 3D world, but is very simple in a 6D world. The coordinate system is a shadow (a projection) of 6D on 3D space. We*

can only see the higher dimensional affair as shadow. But we can live there essentially."[6]

Viewed from overhead one sees the quasi-periodic floor plan in its shapes, and from the side one senses its various dimensions. Although the feeling of such space may initially cause some disorientation, Hizume believes the overall effect will enhance the working of one's mind. His interests, fascination, and passion with forms found in nature, mathematics, music and art all meld and have a profound influence on his ever evolving architectural shapes.



Neuro-architecture (ground floor plan). © 1995 Akio Hizume. ink and pencil on paper 200x300 mm

problem Fibonacci posed in his book *Liber Abaci* in 1202. Mathematicians have repeatedly found this sequence of numbers popping up in nature, art, and music. Each successive number of the sequence is generated by adding the two previous numbers. The golden mean is concealed in the Fibonacci numbers. The ratio of two consecutive Fibonacci numbers get closer and closer to the value of the golden mean; in fact, its limit is the golden mean

1 Starcage wbesite: http://homepage/.nifty.com/starcage/index.html

2 In1992 Hizume invented his 3-dimensional 6-axes self-supporting complex of rods which he named starcage (Japan Patent Pending). At the Bamboo Giant Nursery in Aptos, CA, one of Hizume's bamboo starcages can be seen balancing on stilts high among the bamboo tree tops.

3 Hizume describes MU-MAGARI as is self-complete, self-independent and self-supporting, which can be enlarged so that as it is made wider it becomes more symmetrical.

4 The Hocking Hill State Park, Artists Organization o Columbus, Hocking County Tourism Association, and Ohio University-Lancaster's Wilkes Gallery brought Hizume ot Ohio.

5 Ibid, footnote 1.

6 From personal interview.

19

Fibonacci Sequence

Fibonacci[1], one of the leading mathematicians of the Middle Ages, made contributions to arithmetic, algebra and geometry. He was born Leonardo da Pisa (1175-1250), son of an Italian customs official stationed at Bugia (modern Bougie) in northern Africa. His father's work involved travel to various Eastern and Arabic cities, and it was in these regions that Fibonacci became familiarized with the Hindu-Arabic decimal system, which had place value and used the zero symbol. At this time Roman numerals were still being used for calculating in Italy. Fibonacci saw the value and beauty of the Hindu-Arabic numerals, and was a strong advocate of their use. In 1202 he wrote **Liber Abaci**, a comprehensive handbook explaining how to use the Hindu-Arabic numerals; how addition, subtraction, multiplication and division were performed with these numerals; how to solve problems; and further discussion of algebra and geometry. Italian merchants were reluctant to change their old ways; but through their continual contact with Arabs and the works of Fibonacci and other mathematicians, the Hindu-Arabic system was introduced and slowly accepted in Europe.

Fibonacci sequence — 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

It seems ironic that Fibonacci is famous today because of a sequence of numbers that resulted from one obscure problem in his book, **Liber Abaci**. At the time he wrote the problem it was considered merely a mental exercise. Then, in the 19th century, when the French mathematician Edouard Lucas was editing a four volume work on recreational mathematics, he attached Fibonacci's name to the sequence that was the solution to the problem from **Liber Abaci**. The problem from **Liber Abaci** that generated the Fibonacci sequence is:

[1] Fibonacci literally means son of Bonacci.

1) Suppose a one month old pair of rabbits (male and female) are too young to reproduce, but are mature enough to reproduce when they are two months old. Also assume that every month, starting from the second month, they produce a new pair of rabbits (male & female).

2) If each pair of rabbits reproduces in the same way as the above, how many pairs of rabbits will there be at the beginning of each month?

=pair, mature enough to reproduce

=pair, too young to reproduce

no. of pairs

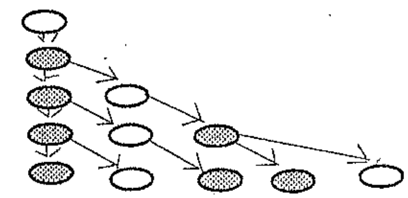
1=F1=1st Fib. no.

1=F2=2nd Fib. no.

2=F3=3rd Fib. no

3=F4=4th Fib. no.

5= F5=5th Fib. no.



Each term of the Fibonacci sequence is the sum of the two preceeding terms and is represented by the formula:

$$F_n=F_{n-1}+F_{n-2}$$

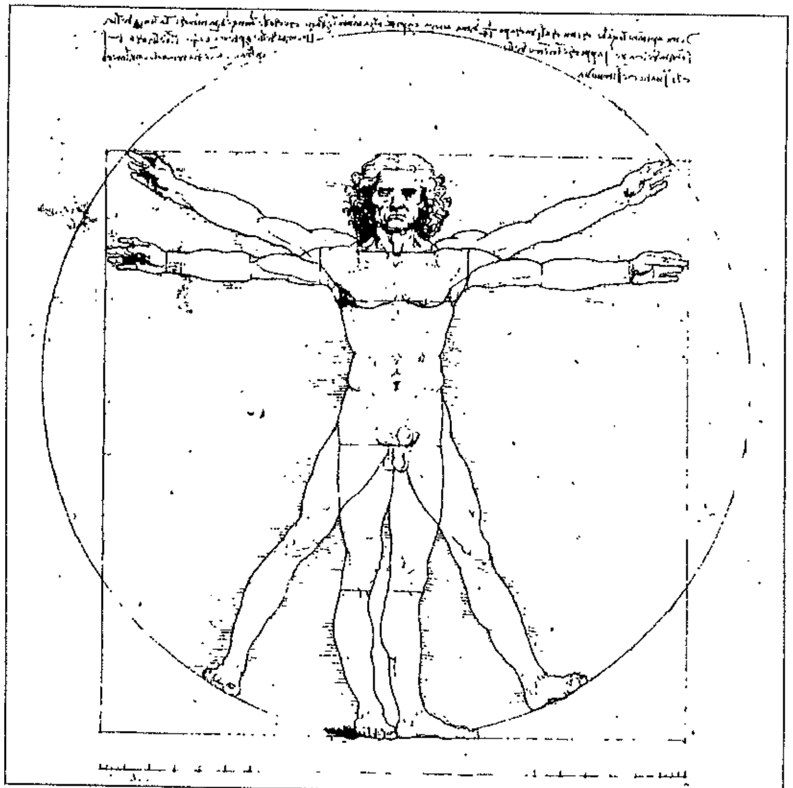
Fibonacci did not study this resulting sequence at the time, and it was not given any real significance until the 19th century when mathematicians became intrigued with the sequence, its properties, and the areas in which it appears.

Fibonacci sequence appears in:

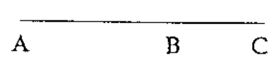
I. *The Pascal triangle, the binomial formula & probability*
II. *the golden ratio and the golden rectangle*
III. *nature and plants*
IV. *intriguing mathematical tricks*
V. *mathematical identities*

Anatomy & the Golden Section

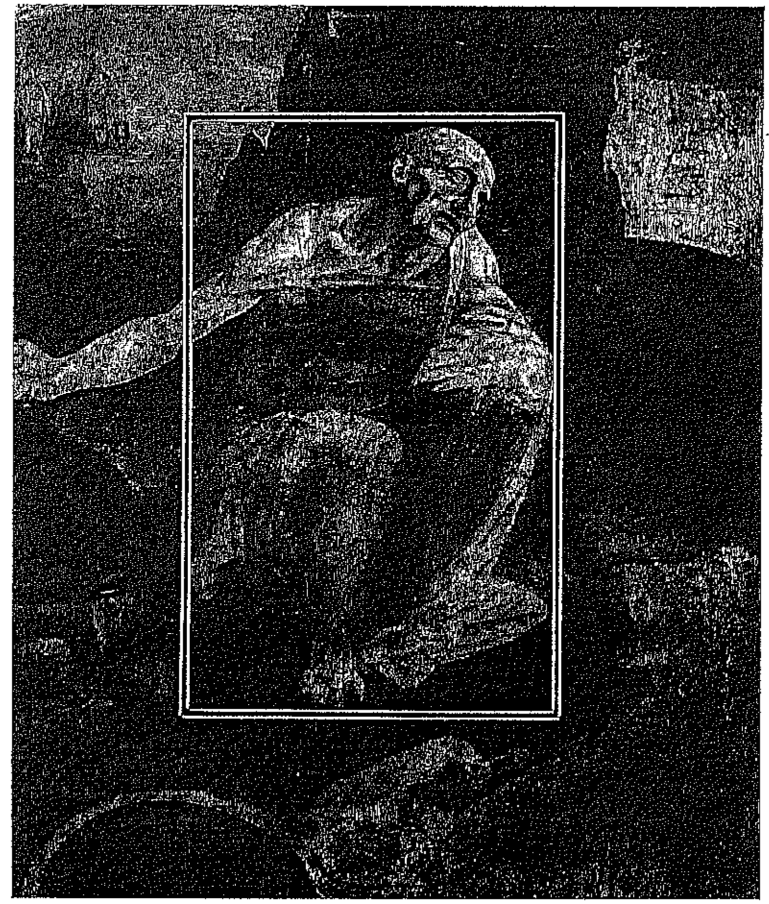
Leonardo da Vinci extensively studied the proportions of the human body. His drawing below has been studied in detail, and shown to illustrate the use of the golden section.[1] This is one of his drawings in the book he illustrated for mathematician Luca Pacioli called *De Divina Proportione* published in 1509.



[1] The term *golden section* is also referred to as the *golden mean, the golden ratio, the golden proportion*. It is the geometric mean when it is located on a given segment as follows. Point B sections off segment AC so that $(|AC|/|AB|)=(|AB|/|BC|)$. The value of the golden mean may be determined as, $\frac{1+\sqrt{5}}{2} \approx 1.6$



The golden section is also present in the unfinished work, *St. Jerome,* by Leonardo da Vinci, painted around 1483. The figure of St. Jerome fits perfectly into a golden rectangle, as superimposed on this drawing. It is believed that this was not an accident, but that Leonardo purposely made the figure conform to the golden section because of his keen interest and use of mathematics in many of his works and ideas. In the words of Leonardo —*"...no human inquiry can be called science unless it pursues its path through mathematical exposition and demonstration."*



St. Jerome. Leonardo da Vinci. Circa 1483

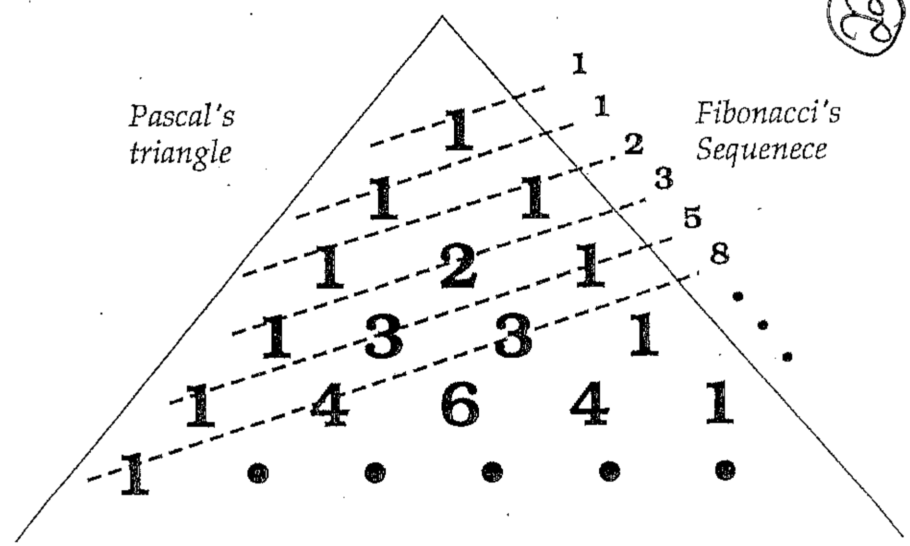
Pascal's triangle, the Fibonacci sequence & binomial formula

Blaise Pascal (1623-1662) was a famous French mathematician who might have become one of the great mathematicians if it were not for his religious beliefs, poor health and unwillingness to exhaust a mathematical topic. His father, fearing that his son would share his keen interest in mathematics[1] and wanting him to develop a broader educational background, initally discouraged him from studying mathematics in order that he might develop other interests. But by the age of twelve Pascal showed such a gift for geometry that his mathematical inclination was thereafter encouraged. He was very talented, and at the age of sixteen he wrote an essay on conics that surprised and astounded mathematicians. In his work was the theorem that came to be known a Pascal's theorem, which states in essence that *opposite sides of a hexagon, which is inscribed in a conic, intersect in three collinear points.* At the age of eighteen he invented one of the first calculating machines. At this time he suffered from poor health, and made a vow to God that he would give up his work in mathematics. But three years later he wrote his work on the *Pascal triangle* and its properties. On the night of November 23, 1654, Pascal had a religious experience that prompted him to devote his life to theology and abandon mathematics and science. Except for one brief period (in 1658-1659), Pascal never studied mathematics again.

Mathematics has a way of connecting ideas that appear unrelated on the surface. So it is with the Pascal triangle, the Fibonacci sequence and Newton's binomial formula. The Pascal triangle, the Fibonacci sequence, and the binomial formula are all interrelated. The design illustrates their relationships. The sums of the numbers along the diagonal segments of the Pascal triangle generate the Fibonacci sequence. Each row of the Pascal triangle represents the coefficients of the binomial (a+b) raised to a particular power.

For example,

$(a+b)^0 = 1$	1
$(a+b)^1 = 1a + 1b$	1 1
$(a+b)^2 = 1a^2 + 2ab + 1b^2$	1 2 1
$(a+b)^3 = 1a^3 + 3a^2b + 3ab^2 + 1b^3$	1 3 3 1



$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n} b^n$$

Newton's binomial formula

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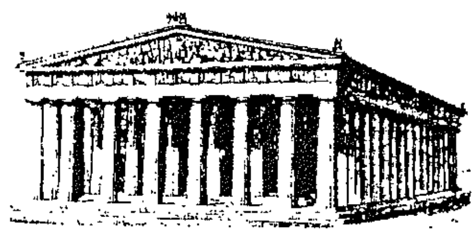
[1]Etienne Pascal, was very much interested in mathematics, and in fact the curve *limaçon of Pascal* is named after him rather than his son.

The Golden Rectangle

The golden rectangle is a very beautiful and exciting mathematical object, which extends beyond the mathematical realm. Found in art, architecture, nature, and even advertising, its popularity is not an accident. Psychological tests have shown the golden rectangle to be one of the rectangles most pleasing to the human eye.

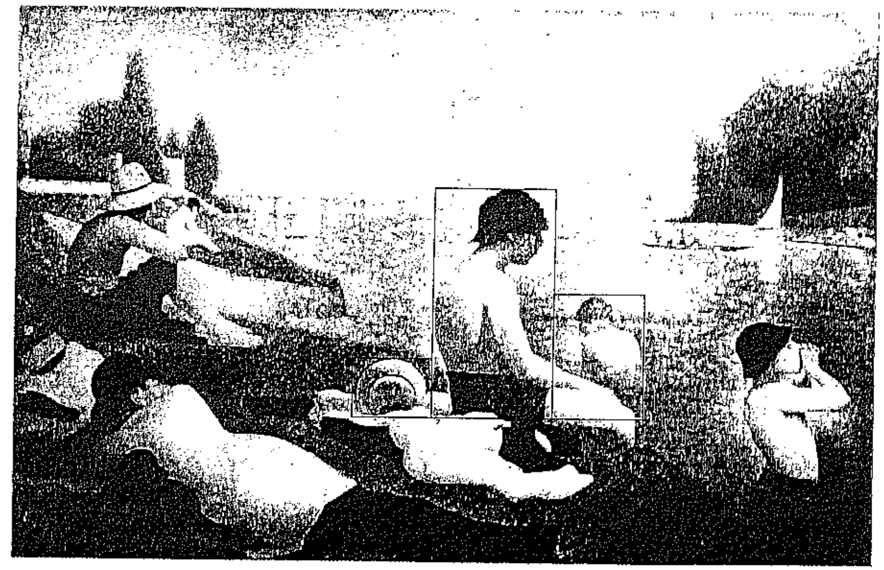
Ancient Greek architects of the 5th century B.C. were aware of its harmonious influence. The Parthenon is an example of the early architectural use of the golden rectangle. The ancient Greeks had knowledge of the golden mean, how to construct it, how to approximate it, and how to use it to construct the golden rectangle. The *golden mean*, ϕ (phi), was not coincidentally the first three letters of *Phidias*, the famous Greek sculptor. Phidias was believed to have used the golden mean and the golden rectangle in his works. The society of Pythagoreans may have chosen the pentagram as a symbol of their order because of its relation to the golden mean.



The Partenon in Athens, Greece.

Besides influencing architecture, the golden rectangle also appears in art. In the 1509 treatise *De Divina Proportione* by Luca Pacioli, Leonardo da Vinci illustrated the golden mean in the make up of the human body. The use of the golden mean in art has come to be labeled as the technique of *dynamic symmetry*.

Albrecht Dürer, George Seurat, Pietter Mondrian, Leonardo da Vinci, Salvador Dali, George Bellows all used the golden rectangle in some of their works to create dynamic symmetry.



Bathers (1859-1891) by French impressionist George Seurat. There are three golden rectangles shown.

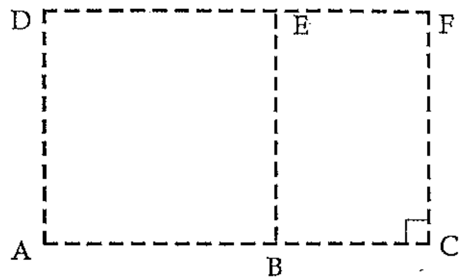
When the geometric mean is located on a given segment, AC, the golden mean[1] is formed, so that

$(|AC|/|AB|)=(|AB|/|BC|)$,
then $|AB|$ is the *golden mean*,
also known as the *golden section*,
the *golden ratio*, or the *golden proportion*.

A B C

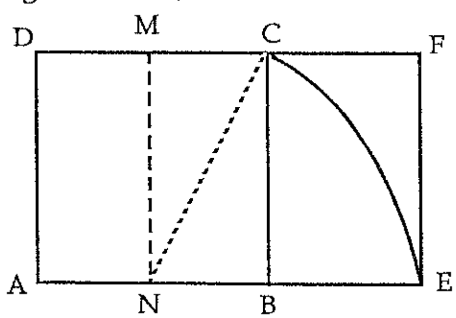
[1] To determine the value of the golden ratio, one must solve the equation $(1/x) = (x/(1-x))$, where $x=|AB|$, $|AC|=1$, and $|BC|=(1-x)$. The golden ratio, $|AC|/|AB|$ or $|AB|/|BC|$ comes out to be — $[(1+\sqrt{5})/2]\approx 1.6$.

Once a segment has been divided into a golden mean, the golden rectangle can easily be constructed as follows:



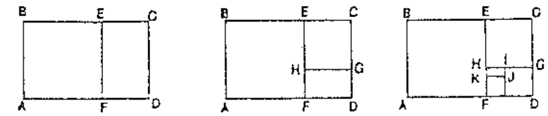
1) Given any segment $\overline{AC}$, with B dividing the segment into a golden mean, construct square ABED.
2) Construct $\overline{CF}$ perpendicular to $\overline{AC}$.
3) Extend ray $\overrightarrow{DE}$ so that line $\overleftrightarrow{DE}$ intersects line $\overleftrightarrow{CF}$ at point F. Then ADFC is a golden rectangle.

A golden rectangle can also be constructed without already having the golden mean, as follows:



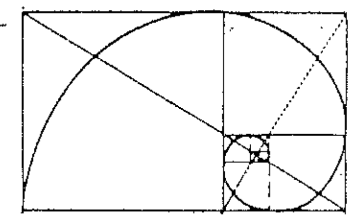
1) Construct any square, ABCD.
2) Bisect the square with segment $\overline{MN}$
3) Using a compass, make arc $\overset{\frown}{EC}$ using center N and radius $|CN|$.
4) Extend ray $\overrightarrow{AB}$ until it intersects the arc at point E.
5) Extend ray $\overrightarrow{DC}$.
6) Construct segment $\overline{EF}$ perpendicular to segment $\overline{AE}$, and ray $\overrightarrow{DC}$ intersects ray $\overrightarrow{EF}$ at point F. Then ADFE is a golden rectangle.

The golden rectangle is also *self-generating*. Starting with golden rectangle ABCD below, golden rectangle ECDF is easily made by drawing square ABEF. Then golden rectangle DGHF is easily formed by drawing square ECGH. This process can be continued indefinitely.



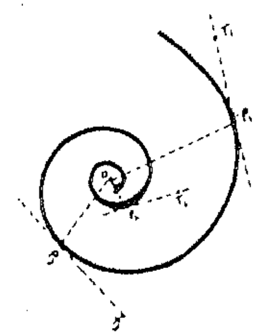
Using the final product of these infinitely many golden rectangles nestled in one another the *equiangular spiral* (also called the *logarithmic spiral)* can be made. Using a compass and the squares of these golden rectangles, make arcs which are quarter circles of these squares. These arcs form the equiangular spiral.

NOTE:
The golden rectangle continually generates other golden rectangles and thus outlines the equiangular spiral. The intersection of the diagonals pictured is the pole or center of the spiral.



O is the center of the spiral.

A radius of the spiral is a segment with endpoints the center O and any point of the spiral.



Notice that each tangent to the point of the spiral forms an angle with that point's radius, e.g. T_1P_1O. The spiral is an equiangular spiral if all such angles are congruent.

This is also called a logarithmic spiral because it increases at a geometric rate, i.e. a power of some number and a power or exponent is another name for logarithm.

The equiangular spiral is the only type of spiral that does not alter its shape as it grows.

Nature has many forms of packaging — squares, hexagons, circles, triangles. The golden rectangle and the equiangular spiral are two of the most aesthetically pleasing forms. Evidence of the equiangular spiral and the golden rectangle are found in starfish, shells, ammonites, the chambered nautilus, seedhead arrangement, pine cones, pineapples, and even the shape of an egg.

Equally exciting is how the golden ratio is linked to the Fibonacci sequence. The limit of the sequence of ratios of consecutive terms of the Fibonacci sequence — (1, 1, 2, 3,. 5, 8, 13, ... , $[F_{n-1}+F_{n-2}]$,...) — is the golden mean, ϕ.

$$\frac{1}{1},\frac{2}{1},\frac{3}{2},\frac{5}{3},\frac{8}{5},\frac{13}{8},\frac{21}{13},\frac{34}{21}\ldots,\frac{F_{n+1}}{F_n}\rightarrow\varphi$$

$$1,\ 2, 1.5, 1.6, 1.625, 1.\overline{615384}, 1.\overline{619047}, \ldots$$

$$\varphi=\frac{1+\sqrt{5}}{2}\approx 1.6$$

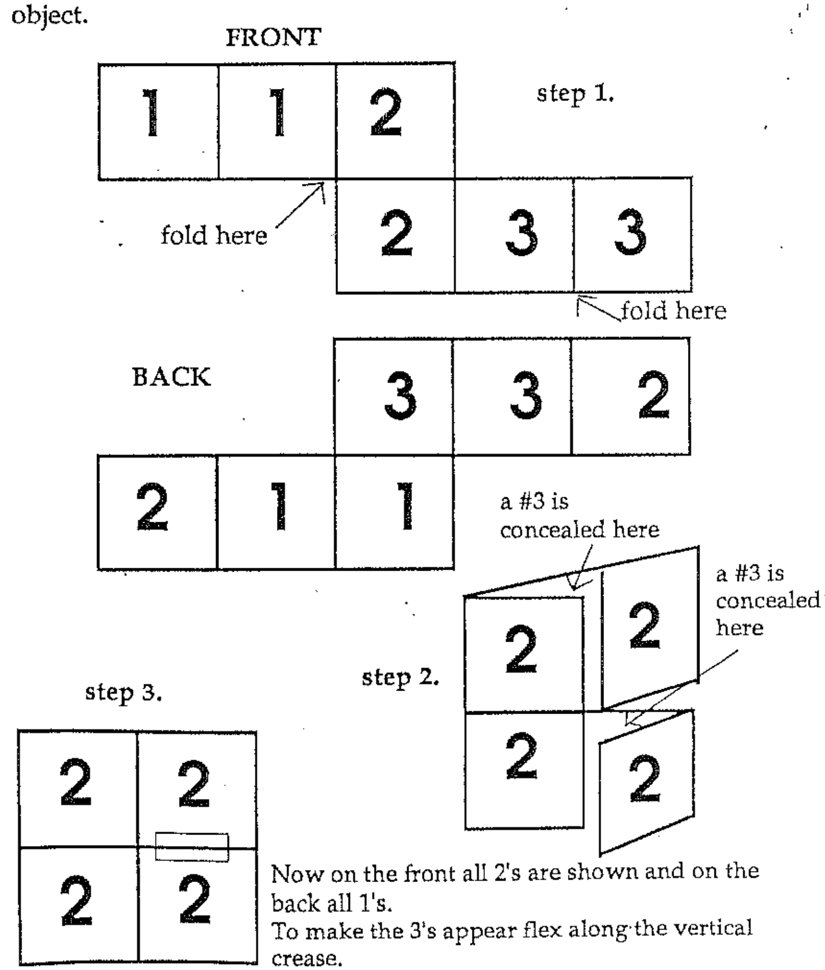
Besides appearing in art, architecture and nature, the golden rectangle is even used today in advertising and merchandising. Many containers are shaped as golden rectangles to possibly appeal to the public's aesthetic point of view. In fact, the standard credit card is nearly a golden rectangle.

Yet the golden rectangle interrelates with other mathematical ideas. Some of these are: infinite series, algebra, an inscribed regular decagon, Platonic solids, equiangular and logarithmic spirals, limits, the golden triangle, and the pentagram.

Making a Tri-Tetra Flexagon

In a broad sense flexagons can be considered a type of topological puzzle. They are figures made from a sheet of paper, but end up having a varying number of faces which are brought to view by a series of flexings.

The object below is a called a tri-tetra flexagon. Tri stands for the number of faces and **tetra** for the number of sides of the object.



Now on the front all 2's are shown and on the back all 1's.
To make the 3's appear flex along the vertical crease.